Numerical errors in weight vector computation

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Weight vector computation problem

Minimize array output in all directions but that of the target direction d

Minimization problem (1):

[redacted] C is signal covariance matrix

Solution: $w = \dfrac{C^{-1}d}{d^H C^{-1} d}$

C is approximated by

$A = \sum S_k S_k^H = X^H X$ where $X = [S_1, S_2, \ldots, S_K]$

Minimization problem (2):

[redacted] A is sample covariance matrix

Solution: $w = \dfrac{A^{-1}d}{d^H A^{-1} d}$

$A = X^H X$. Minimization problem (3):

[redacted] constrained least squares

Algorithms for (2)

$A = X^H X > 0$, need to compute $A^{-1}d$

Normal Equations (NE)		Semi NE
(I) Cholesky	(II) GE	(III) QR
1) $A = U^H U$	1) $A = L^H U$	1) $X = QR$
2) $U^H u = d$	2) $L^H u = d$	2) $R^H u = d$
3) $U v = u$	3) $U v = u$	3) $R v = u$
4) $w = v/(d^H v)$	4) $w = v/(d^H v)$	4) $w = v/(d^H v)$

Computed A may become indefinite
so use GE instead of Cholesky

Algorithms for (3) – Null Space Method

1) $Q^H \begin{bmatrix} d^H \\ X \end{bmatrix} H = \begin{bmatrix} 0 & 0 \\ e_1{}^T & 0 \\ L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$ Generalized QL decomposition

2) $L_{22}w_2 = - L_{21}$ (L_{22} is better conditioned than X)

3) $w = H\,[\,1\ w_2{}^H]^H$

4) $||Xw|| = |L_{11}|$ (L_{11} is the norm of the residual)

Sensitivity Analysis - NE

A replaced by $\widehat{A} = A + \Delta A$, $||\Delta A|| < \varepsilon \, ||A||$

$$w = \frac{A^{-1}d}{d^H A^{-1} d} \qquad \widehat{w} = \frac{\widehat{A}^{-1}d}{d^H \widehat{A}^{-1} d}$$

Then

[redacted]

Sensitivity Analysis - SNE

X replaced by $\widehat{X} = X + \Delta X$, $\|\Delta X\| < \varepsilon \|X\|$

$$w = \frac{(X^H X)^{-1} d}{d^H (X^H X)^{-1} d} \qquad \widehat{w} = \frac{(\widehat{X}^H \widehat{X})^{-1} d}{d^H (\widehat{X}^H \widehat{X})^{-1} d}$$

[redacted]

can be small

Sensitivity Analysis - NS

X replaced by $\widehat{X} = X + \Delta X$, $\|\Delta X\| < \varepsilon \, \|X\|$

$$\min_{w^H d = 1} \|Xw\|_2 \qquad\qquad \min_{\hat{w}^H d = 1} \|\widehat{X}\,\hat{w}\|_2$$

[redacted]

SNE&NS - Small residual case

$X = W\Sigma V^H$, $\Sigma = \mathrm{diag}(\sigma i)$, $V = [v_1, \ldots, v_n]$

If $d = v_n$ then $w = v_n$, $||Xw|| = \sigma_n$ and

[redacted]

Conclusions

- SNE and NS are equally accurate
  - this is not the case for general LS problems
- If $||Xw|| = \sigma_n$ then SNE and SN are more accurate than NE
  - in NE use GE instead of Cholesky
- If $||Xw|| = \sigma_1$ then NE, SNE and SN are equal
  - NE is the least expensive
- cond number determined by submatrices of L
  - cond number can be small even if that of X is large